Let a be positive. On ON lay off $OA_1=a$. At A_1 erect a perpendicular cutting OA in A_1'. By similar triangles, $A_1A_1'=a^2/x$. On ON lay off $OA_2=A_1A_1'$, and erect a perpendicular cutting OA in A_2'. Then $A_2A_2'=a^3/x^2$; and so on, till we obtain $A_{\alpha-1}A'_{\alpha-1}=a^\alpha/x^{\alpha-1}$. If β is positive, lay of, on ON, $OB_1=A_{\alpha-1}A_{\alpha-1}'$ (or equal to NA if $\alpha=1$). Erect a perpendicular cutting OB in B_1'; and so on, till we obtain $B_\beta B_\beta'=\dfrac{a^\alpha b^\beta}{x^{\alpha+\beta-1}}$. If γ, for example, is negative, draw through B_β' a parallel to ON cutting OC in C_1'. From C_1' drop a perpendicular on ON cutting ON in C_1. Then $OC_1=\dfrac{a^\alpha b^\beta c^{-1}}{x^{\alpha+\beta-2}}$. On NC lay off $NC_2''=OC_1$. Erect a perpendicular cutting OC_2 in C_2'. Drop a perpendicular C_2C_2' on ON. Then $OC_2=\dfrac{a^\alpha b^\beta c^{-2}}{x^{\alpha+\beta-3}}$. Continuing in this way, we obtain OL_λ or $L_\lambda L_\lambda'=\dfrac{a^\alpha b^\beta \ldots. l^\lambda}{x^{\alpha+\beta\ldots.\lambda-1}}=\dfrac{a^\alpha b^\beta \ldots. l^\lambda}{x}$. By laying off OL_λ in succession P times and applying the known construction for the Q section of a line we obtain $SR=\dfrac{P}{Q}\dfrac{a^\alpha b^\beta \ldots. l^\lambda}{x}$.

Making a similar construction for each of the terms under the radical considered and taking the algebraic sum of the resulting lines we obtain $TU=\dfrac{\Sigma(P/Q)a^\alpha b^\beta \ldots. l^\lambda}{x}$.

From N lay off on ON, $NV=TU$,—away from O if TU is positive. On OV as a diameter construct a circle cutting NA in M (and M'). Then $NM=\sqrt{\Sigma \dfrac{P}{Q}a^\alpha b^\beta \ldots. l^\lambda}=m$ (say).

[We may consider $NM'=-NM$. In this way the conjugate values of y may be constructed].

Hence $y=\varphi_1(a, b, \ldots., l, m)$, where φ_1 is homogeneous and contains one less radical than φ. Continuing in this way, we obtain finally $y=\psi(a, \ldots., l, m, \ldots., r)$, where ψ is a sum of terms of the form $(P'/Q')a^{\alpha'}b^{\beta'}\ldots.r^{\rho'}$ and $\alpha'+\beta'\ldots.+\rho'=1$. Constructing the sum of these terms after the manner above indicated, we obtain the length y required.

A GENERAL THEORY OF PROJECTILES.

By M. E. GRABER, Heidelberg University. Tiffin. Ohio.

There seems to be a lack of uniformity in defining the term *projectile.* Some definitions are too exclusive or special, while others are too inclusive. From a consideration of the conditions involved the following definition seems to answer the purpose well:

A projectile is any body projected from a given point in a given direction subject

to the force of gravity and whatever other forces are inherent in the medium through which the body is projected.

This definition, not being limited to one medium, enables us to apply the term "projectile" not only to particles impelled through the atmosphere but also to particles projected through any medium. This would of course include submarine torpedoes in the category of projectiles.

The method of treatment of the theory of projectiles has generally been first the discussion of motion *in vacuo* and then the discussion of motion in a resisting medium such as the atmosphere. Why not first deduce and discuss formulae of general application to the motion of particles and then consider motion *in vacuo* as a special case with particular values assigned to the different coefficients and factors?

The first consideration in the discussion of the general case is the resistance of the air to the motion of a particle through the atmosphere.

It has been proven by experiment that the resistance of the air encountered by a projectile is proportional to the exposed area. This area is proportional to the square of the diameter of the projectile. Calling this resistance R, we have $R=d^2\phi(v)$, and the corresponding retardation $r=\frac{g}{w}R=g\frac{d^2}{w}\phi(v)$. Putting D (ballistic coefficient) for $\frac{w}{d^2}$, we have $r=\frac{g}{D}\phi(v)$. Assuming that $\phi(v)=B_1v^n$, in which B_1 and n are constants, to be determined by experiment, we get

$$R=\frac{B_1d^2v^n}{g} \text{ and } r=\frac{B_1v^n}{D}\text{....(1)}.$$

Assuming that the axis of the (oblong) projectile lies constantly in the tangent to the trajectory and that the air is calm and of uniform density, the retardation along the tangent at any point of the trajectory due to air resistance is $\frac{B_1v^n}{D}$; the retardation due to grauity is $g\sin\theta$. Consequently $-\frac{dv}{dt}=\frac{B_1v^n}{D}+g\sin\theta$. The velocities parallel to the X-axis, which is horizontal forwards, and the Y-axis, which is vertical upwards, are $v\cos\theta$ and $v\sin\theta$. The corresponding retardations are $\frac{1}{D}B_1v^n\cos\theta$ and $\frac{1}{D}B_1v^n\sin\theta+g$. Therefore

$$\frac{d(v\cos\theta)}{dt}=-\frac{B_1v^n}{D}\cos\theta\text{....(2)},$$

$$\frac{d(v\sin\theta)}{dt}=-\left(\frac{B_1v^n\sin\theta}{D}+g\right)\text{....(3)}.$$

Performing the indicated differentiations and comparing the resulting equations we get

$$\frac{vd\theta}{dt}=-g\cos\theta....(4),$$

an expression for the resultant of the forces normal to the direction of resistance. Again, represent the horizontal velocity by v_1; then of course $v_1=v\cos\theta$. Substituting this in (2) and (4), we get

$$\frac{dv_1}{dt}=-\frac{B_1 v_1{}^n}{D\cos^{n-1}\theta}....(5).$$

$$\frac{v_1 d\theta}{dt}=-g\cos^2\theta....(6).$$

The relations between elements of time and elements of the trajectory at any point are given by the following: $dx=v_1 dt$, $dy=v_1\tan\theta dt$, $ds=v_1\sec\theta dt$, where s is the length of the trajectory from the origin. In the above three equations substituting for dt its value $-\frac{v_1}{g}\sec^2\theta d\theta$, we get

$$dx=-\frac{v_1^{\,2}}{g}\sec^2\theta d\theta....(7),$$

$$dy=-\frac{v^2}{g}\tan\theta\sec^2\theta d\theta....(8),$$

$$ds=-\frac{v_1^{\,2}}{g}\sec^3\theta d\theta....(9).$$

We have now discussed the general theory sufficiently to enable us to derive all the equations of the trajectory *in vacuo*. *In vacuo* the resistance being 0, equation (5) reduces to $dv_1=0$. Consequently v_1 (horizontal velocity) is constant and equal to initial velocity V. Therefore $v\cos\theta=V\cos\phi$. Keeping in mind that ϕ is the initial value of θ, and θ the angle which the tangent of the trajectory at the point x, y makes with the axis of abscissae. Integrating $dt=-\frac{v_1}{g}\sec^2\theta d\theta$ between the limits ϕ and θ, we get

$$t=\frac{V_1}{g}(\tan\phi-\tan\theta)....(10),$$

if $v_1=V_1$. Likewise, by integrating (7), (8), and (9) under the same conditions, we get

$$x=\frac{V_1^2}{g}(\tan\phi-\tan\theta)....(11),$$

$$y=\frac{V_1^2}{2g}(\tan^2\phi-\tan^2\theta)....(12),$$

$$S=\frac{V_1^2}{g}[(\phi)-(\theta)],$$

where $\theta=\int\frac{d\theta}{\cos^3\theta}$ and $\phi=\int\frac{d\phi}{\cos^3\phi}$, both integrals having the same lower limits.

To derive the equation of the trajectory *in vacuo*, eliminate $\tan\theta$ from (11) and (12) by division and addition and we have

$$y=x\tan\phi-\frac{\frac{1}{2}gx^2}{V_1^2},$$

the equation of a parabola whose axis is vertical.

To determine the range, we merely substitute X for x and $-\phi$ for θ in (11); whence $X=\frac{2V_1^2\tan\phi}{g}$. Then since $V\cos\phi=V_1$, $X=\frac{V^2\sin2\phi}{g}$.

To determine the time of flight, set $\theta=-\phi$ in (10). We have

$$T=\frac{2V_1\tan\phi}{g}=\frac{2V\sin\phi}{g}.$$

To determine velocity, since $V\cos\phi=v\cos\theta=V_1$, we have from (12),

$$v^2\sin^2\theta=V^2\sin^2\phi-2gy.$$

Adding $v^2\cos^2\theta$ to the first member and $V^2\cos^2\phi$ (its equal) to the second member, we get $v^2=V^2-2gy$. In a similar manner, we can determine all the properties of a trajectory *in vacuo*.

A PEDAGOGICAL QUESTION IN SPHERICAL TRIGONOMETRY.

By G. W. GREENWOOD, Professor of Mathematics and Astronomy in McKendree College.

The numerous and seemingly disconnected formulae of right and quadrantal spherical triangles are often found, both by student and instructors, difficult to remember and thus detract from the study of spherical trigonometry.

I have found the application of the general formulae for oblique triangles to these triangles very easy and useful, the necessary sets of formulae being

$$\cos a=\cos b\cos c+\sin b\sin c\cos A\ldots.(1),$$

$$-\cos A=\cos B\cos C-\sin B\sin C\cos a\ldots.(2),$$

$$\cot a\sin b=\cos b\cos C+\sin C\cot A\ldots.(3),$$

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}\ldots.(4).$$